Classification of competitiveness types using copula

Cezar Mereuta and Lucian-Liviu Albu and Daniel Ciuiu

Romanian Center for Economic Modeling, Romanian Institute for Economic Forecasting, Technical University of Civil Engineering, Bucharest, Romania; Romanian Institute for Economic Forecasting

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Cezar Mereuță
Romanian Center for Economic Modeling (CERME), Bucharest, Romania
e-mail: cezar_mereuta@yahoo.com
Lucian Liviu Albu
Romanian Institute for Economic Forecasting
e-mail: albul@ipe.ro
Daniel Ciuiu
Technical University of Civil Engineering, Bucharest;
Romanian Institute for Economic Forecasting
e-mail: dciuiu@yahoo.com

Abstract

In this paper we classify competitive markets using a new form of normalized Herfindahl index and the degree of dominance of the leader. For this purpose we use the notion of copula, which connects two or more random variables with given marginals.

The parameters of the two marginals (which are supposed to be normal) are estimated by the moments' method, and the parameter of the copula is computed using the value τ of Kendall.

1. Introduction

In [7] there is defined the market share of the company *i* by the formula

$$Cp_i = \frac{CA_i}{\sum_{j=1}^{n} CA_j} = \frac{CA_i}{CA_T}, \tag{1}$$

where CA_i is the benefit of the company *i*. We denote next by $p_i = Cp_i$, and we reorder the companies such that $p_1 \geq ... \geq p_n$. In this case p_1 is the weight of the leader (see [7]).

The Herfindahl index, or the informational energy of Onicescu is (see [9,10,7,11])

$$H = \sum_{i=1}^{n} p_i^2 . \tag{2}$$

In [7] there are considered 553 clustered markets, 235 in 2004 and 318 in 2008 as follows

1) In the year 2004:
   a) 174 markets clustered CAEN Rev. 1 at group level (three digits)
   b) 47 markets clustered CAEN Rev. 1 at division level (two digits)
   c) 13 markets clustered CAEN Rev. 1 at section level (one alphabetic character)
   d) one national system
2) In the year 2008:
   a) 218 markets clustered CAEN Rev. 2 at group level (three digits)
   b) 80 markets clustered CAEN Rev. 2 at division level (two digits)
   c) 19 markets clustered CAEN Rev. 2 at section level (one alphabetic character)
   d) one national system

If we denote by *n* the number of companies and by p_1 the weight of the leader we obtain the regression

$$\begin{array}{llll} \log H = & 1.239375 \log p_1 & -0.163945 \log n & +0.164457 \\ & [0.016777] & [0.008147] & [0.016634] \end{array}$$

(3)

with the determination coefficient $R^2 = 0.97060142$ and the estimated standard deviation 0.10470.

Becausethe Herfindahl index has a high variation degree (in the above case the ratio between maximum and minimum is 832.2702) Mereuță (see [7]) introduced the normalized Herfindahl index:

$$M = \frac{\ln H + \ln n}{\ln n}. \tag{4}$$

The above cohesion measure is the normalized quadratic Rényi entropy, where the quadratic Rényi entropy is $R = -\ln H$.

Another parameter used to measure the cohesion of the market shares is the degree of dominance of the leader (see [7])

$$Gdl = \frac{\frac{p_1^2}{H} - \frac{1}{n}}{1 - \frac{1}{n}}. \tag{5}$$

Noticing that $0 \le M \le 1$ and $0 \le Gdl \le 1$ Mereuță defines the matrix of cohesion degrees. There are obtained nine regions of the unit square using the lines $Gdl = 0.4$, $Gdl = 0.6$, $M = 0.4$ and $M = 0.6$.

Definition 1 ([12,8,13]) *A copula is a function* $C : [0,1]^n \to [0,1]$ *such that*

1) *If there exists* i *such that* $x_i = 0$ *then* $C(x_1,...,x_n) = 0$.
2) *If* $x_j = 1$ *for all* $j \ne i$ *then* $C(x_1,...,x_n) = x_i$.
3) C *is increasing in each argument.*

We have the following theorem (see [12,8,13]).

Theorem 1 (Sklar) *Let* X_1, X_2,..., X_n *be random variables with the cumulative distribution functions* F_1, F_2,..., F_n, *and the common cdf* $H(x_1,...,x_n) = P(X_1 \le x_1,...,X_n \le x_n)$. *In this case there exists a copula* $C(u_1,...,u_n)$ *such that* $H(x_1,...,x_n) = C(F_1(x_1),..., F_n(x_n))$. *The copula* C *is well defined on the chartesian product of the images of the marginals* F_1, F_2,..., F_n.

Definition 2 ([12,14,15]) *If* $n = 2$ *the copula* C *is Archimedean if* $C(u,u) < u$ *for any* $u \in (0,1)$ *and* $C(C(u,v),w) = C(u,C(v,w))$ *for any* $u,v,w \in [0,1]$. *If* $n > 2$ *the copula* C *is Archimedean if there exists a* $n-1$ *Archimedean copula* C_1 *and a* $2-$*Archimedean copula* C_2 *such that* $C(u_1,...,u_n) = C_2(C_1(u_1,...,u_{n-1}),u_n)$.

Consider a function $\varphi : (0,1] \to R$ decreasing and convex with $\varphi(1) = 0$ and its pseudo-inverse g ($g(y)$ has the value x if there exists x such that $\varphi(x) = y$ and 0 in the contrary case). We know (see [5,12]) that a copula C is Archimedean if and only if there exists a function φ as above such that for any $x, y \in [0,1]$ we have

$$C(x, y) = g(\varphi(x) + \varphi(y)). \tag{6}$$

In [14,15] there are presented methods to simulate Archimedean copulas, and in [2] there are presented algorithms to simulate queueing systems with one channel with arrivals and services depending through copulas.

In [4] there are found analytical formulae for the copulas that connect the number of customers in a Gordon and Newell queueing network, and their corresponding Spearman ρ and Kendall τ. This value is (see [8]):

$$\tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0) =$$
$$4\int_0^1\int_0^1 C(u, v)\tfrac{\partial^2 C}{\partial u \partial v}\, dudv - 1 = 1 - 4\int_0^1\int_0^1 \tfrac{\partial C}{\partial u} \cdot \tfrac{\partial C}{\partial v}\, dudv.$$

Sometimes we need the overlay probabilities, and we need in this case the notion of co-copula (see [14])

$$C^*(u_1,...,u_n) = C(1-u_1,...,1-u_n) + \sum_{i=1}^{n} u_i - n + 1. \tag{8}$$

The probabilistic interpretation of the co-copula is that if X_1,..., X_n are random variables having the marginals $F_1,..., F_n$ and they are connected by the copula C, we have

$$\overline{H}(x_1,...,x_n) = P(X_1 \geq x_1,..., X_n \geq x_n) = C^*\left(\overline{F_1}(x_1),..., \overline{F_n}(x_n)\right), \qquad (8')$$

where $\overline{F_i}(x_i) = 1 - F_i(x_i)$.

2. The new matrix of cohesion degrees using isolines

In the matrix of cohesion degrees defined by Mereuță (see [7]) the regions are separated by $X = \alpha$, or by $Y = \beta$. The regions created by these lines are

$$\begin{cases} P(X \leq \alpha) = F(\alpha) \\ P(X \geq \alpha) = 1 - F(\alpha) \end{cases}, \text{ respectively} \qquad (9)$$

$$\begin{cases} P(Y \leq \beta) = G(\beta) \\ P(Y \geq \beta) = 1 - G(\beta) \end{cases}, \qquad (9')$$

where F and G are the marginals of the Gdl (first axis) and M (second axis). But they do not take into account on the relation between the random variables Gdl and M.

Suppose that the above random variables have normal marginals, as in [7], but they are connected by the copula C. The marginal parameters are estimated using the moments' method, and the parameter θ of the copula C is estimated as follows.

First we estimate τ using the empirical probabilities in the above formula, and next we compute the last term: we find τ in function of θ. For instance, in the case of Farlie-Gumbel-Morgestern copula (see [8,12]) we find

$$\tau = \frac{2\theta}{9}, \text{ and from here} \qquad (10)$$

$$\theta = \frac{9\tau}{2}. \qquad (10')$$

For the Fréchet family the copula is a mixture between the upper Fréchet bound, $\min$ and the copula product (the independence case) with the weights θ, respectively $1-\theta$. Due to the fact that in the $\min$ case we have $\tau = 1$, and in the product case we have $\tau = 0$ we obtain

$$\theta = \tau. \qquad (11)$$

When the copula is Archimedean and we know the function φ in (6) we use the variables change $x = \varphi(u)$ and $y = \varphi(v)$, and finally we obtain

$$\tau = 1 - 4 \cdot \int_0^{\varphi(0)} \int_0^{\varphi(0)} \left(g'(x+y)\right)^2 dxdy. \qquad (7')$$

In the case of Clayton family we have

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}. \qquad (12)$$

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -u^{-\theta-1}$, and from here

$$\varphi(u) = \frac{u^{-\theta} - 1}{\theta}, \text{ and} \qquad (12')$$

$$g(w) = (\theta w + 1)^{-\frac{1}{\theta}}. \qquad (12'')$$

Using (7') we obtain

$$\tau = \frac{\theta}{\theta + 2}, \text{ and from here} \qquad (13)$$

$$\theta = \frac{2 \cdot \tau}{1 - \tau}. \qquad (13')$$

Other family of Archimedean copulas presented in [5,6,8] and simulated in [2] is the Frank family. In this case for $\theta \in \mathrm{R}^*$ we have

$$C(u,v) = -\frac{1}{\theta} \cdot \ln\left(\frac{e^{-\theta(u+v)} - e^{-\theta u} - e^{-\theta v} + e^{-\theta}}{e^{-\theta} - 1} \right). \tag{14}$$

We obtain also the copula *Prod* for $\theta = 0$ and the copula $\min$ for $\theta \to \infty$. For $\theta \to -\infty$ we obtain the lower Fréchet bound W.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = \frac{\theta e^{-\theta u}}{e^{-\theta u} - 1}$, and from here

$$\varphi(u) = \ln \frac{1 - e^{-\theta}}{1 - e^{-\theta \cdot u}}, \text{ and} \tag{14'}$$

$$g(w) = -\frac{1}{\theta} \ln\left(\gamma e^{-w} + 1\right), \ where\ \gamma = e^{-\theta} - 1. \tag{14''}$$

For this family we obtain

$$\tau = 1 - 4 \cdot I, \text{ where} \tag{15}$$

$$I = \frac{1}{\ln^2(1+\gamma)} \cdot \int_0^{\gamma} \frac{\ln(1+x)}{x} + \frac{1}{1+x} dx. \tag{15'}$$

In the case $\theta \neq 0$ we multiply the relation (18') by $\ln^2(1+\gamma)$, and in the case $\theta = \gamma = \tau = 0$ and $I = \frac{1}{4}$ we compute $I'(0) = \frac{1}{36}$ using the Taylor series for $\ln(1+x)$ and $\frac{1}{1+x}$. We obtain the Cauchy problem

$$\begin{cases} \gamma'(I) = \dfrac{\ln^2(1+\gamma(I))}{\frac{\ln(1+\gamma(I))}{\gamma(I)} - \frac{1}{1+\gamma(I)} + \frac{2 \cdot I \cdot \ln(1+\gamma(I))}{1+\gamma(I)}} \ for\ I \neq \frac{1}{4} \\ \gamma'\left(\frac{1}{4}\right) = 36 \\ \gamma\left(\frac{1}{4}\right) = 0 \end{cases}.$$

Because $I = \frac{1-\tau}{4}$ we obtain the Cauchy problem

$$\begin{cases} \gamma'(\tau) = \dfrac{\ln^2(1+\gamma(\tau))}{4 \cdot \frac{\ln(1+\gamma(\tau))}{\gamma(\tau)} - \frac{4}{1+\gamma(\tau)} + \frac{2 \cdot (1-\tau) \cdot \ln(1+\gamma(\tau))}{1+\gamma(\tau)}} \ for\ \tau \neq 0 \\ \gamma'(0) = -9 \\ \gamma(0) = 0 \end{cases}.$$

Finally we take into account that $\gamma(\theta) = e^{-\theta} - 1$ and $\gamma'(\tau) = -e^{-\theta} \cdot \theta'(\tau)$. We obtain

$$\begin{cases} \theta'(\tau) = \dfrac{\theta^2}{2 \cdot (1-\tau) + 4 - \frac{4 \cdot \theta}{e^{\theta} - 1}} \ for\ \tau \neq 0 \\ \theta'(0) = 9 \\ \theta(0) = 0 \end{cases}. \tag{16}$$

The above Cauchy problem is solved using the Runge-Kutta method.

In the case of the Gumbel-Hougaard family (see [5,8,13]) we have for $\theta \geq 1$ and $\beta = \frac{1}{\theta}$

$$C(u,v) = e^{-\left((-\ln u)^{\theta} + (-\ln v)^{\theta}\right)^{\beta}}. \tag{17}$$

For $\theta = 1$ we obtain the copula *Prod* and for $\theta \to \infty$ we obtain the copula $\min$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{\theta(-\ln u)^{\theta-1}}{u}$, and from here

$$\varphi(u) = (-\ln u)^{\theta}, \text{ and} \tag{17'}$$

$$g(x) = e^{-x^{\beta}}. \tag{17''}$$

For this family we obtain

$$\tau = 1 - \frac{1}{\theta}, \text{ and from here} \tag{18}$$

$$\theta = \frac{1}{1-\tau}. \quad (18')$$

The Gumbel-Barnett copula is

$$C(u,v) = u \cdot v \cdot e^{-(\theta(\ln u)(\ln v))}, \text{ with } 0 < \theta \le 1. \quad (19)$$

We notice that we have also the copula product (independence) for $\theta \to 0$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u(1-\theta \ln u)}$, and from here

$$\varphi(u) = \frac{\ln(1-\theta \ln u)}{\theta}, \text{ and} \quad (19')$$

$$g(x) = e^{\frac{1-e^{\theta x}}{\theta}}. \quad (19'')$$

Using (7') we obtain

$$\tau = -e^{\beta} \cdot \int_{\beta}^{\infty} \frac{e^{-x}}{x} dx < 0. \quad (20)$$

where $\beta = \frac{2}{\theta}$.

The Ali-Mikhail-Haq copula is

$$C(u,v) = \frac{u \cdot v}{1-\theta(1-u)(1-v)}, \text{ with } -1 \le \theta \le 1. \quad (21)$$

We notice that we have the copula *Prod* (independence) for $\theta = 0$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u(1-\theta(1-u))}$, and from here

$$\varphi(u) = \frac{1}{1-\theta} \cdot \ln\left(\theta + \frac{1-\theta}{u}\right), \text{ and} \quad (21')$$

$$g(x) = \frac{1-\theta}{e^{(1-\theta)x} - \theta}. \quad (21'')$$

Using (7') we obtain

$$\tau = 1 - \frac{2(1-\theta)^2 \ln(1-\theta)}{3\theta^2} - \frac{2}{3\theta}. \quad (22)$$

In the above formula τ is increasing on θ, and we have $\tau(-1) = \frac{5-8\ln 2}{3}$ and $\tau(1) = \frac{1}{3}$. If we know τ we obtain θ using the bisection method.

Definition 3 *Let (X,Y) be a bi-variate random variable such that the random variables X and Y are connected by the copula C.*

The copula of non-overlay, non-overlay for (X,Y) is $C_{11}(u,v) = C(u,v)$.

The copula of overlay, overlay for (X,Y) is $C_{00}(u,v) = C^*(1-u,1-v)$.

The copula of non-overlay, overlay for (X,Y) is $C_{10}(u,v) = u - C(u,v)$.

The copula of overlay, non-overlay for (X,Y) is $C_{01}(u,v) = v - C(u,v)$.

Remark 1 *If the marginal distributions are uniform on $[0,1]$ then, if we denote by H the common cumulative distribution function, we have: $C_{11}(u,v) = H(u,v) = P(X \le u, Y \le v)$, $C_{00}(u,v) = \overline{H}(u,v) = P(X \ge u, Y \ge v)$, $C_{10}(u,v) = P(X \le u, Y \ge v)$ and $C_{01}(u,v) = P(X \ge u, Y \le v)$. For other marginal distributions, F and respectively G, we have $P(X \le x, Y \le y) = C_{11}(F(x), G(y))$, $P(X \ge x, Y \ge y) = C_{00}(F(x), G(y))$, $P(X \le x, Y \ge y) = C_{10}(F(x), G(y))$ and $P(X \ge x, Y \le y) = C_{01}(F(x), G(y))$.*

We will find the isolines in u,v given by $\alpha_1, \ldots, \alpha_k$ with $0 < \alpha_1 < \ldots < \alpha_k$

$$\begin{cases} C_{11}(u,v) = \alpha_j \\ C_{00}(u,v) = \alpha_j \\ C_{10}(u,v) = \alpha_j \\ C_{01}(u,v) = \alpha_j \end{cases}, \text{ where } j = \overline{1,k} \tag{23}$$

The corresponding isolines in x, y are built from the isolines in u, v such that $x = F^{-1}(u)$ and $y = F^{-1}(v)$. These are the separators of the regions bordered by $x_{\min} = \min\limits_{i=\overline{1,n}} X_i$, $x_{\max} = \max\limits_{i=\overline{1,n}} X_i$, $y_{\min} = \min\limits_{i=\overline{1,n}} Y_i$ and $y_{\max} = \max\limits_{i=\overline{1,n}} Y_i$. The above regions have corresponding regions in the plane Ouv in the box bordered by $u_{\min} = F(x_{\min})$, $u_{\max} = F(x_{\max})$, $v_{\min} = G(y_{\min})$ and $v_{\max} = G(y_{\max})$.

3. Application

Consider the above 553 clustered markets (see [7]). We have $Gdl_{\min} = 0.074$, $Gdl_{\max} = 0.9997$, $M_{\min} = 0.1954$, $M_{\max} = 0.9769$. In [7] the marginal distributions are considered normal. Using the moments method we obtain $\hat{\mu}_{Gdl} = \overline{Gdl} = 0.47432$, $\hat{\sigma}^2_{Gdl} = S^2_{Gdl} = 0.05801$, $\hat{\mu}_M = \overline{M} = 0.51181$ and $\hat{\sigma}^2_M = S^2_M = 0.01745$. From these estimated parameters we obtain the box in the plane Ouv $u_{\min} = 0.0067$, $u_{\max} = 0.9988$, $v_{\min} = 0.0002$ and $v_{\max} = 0.9989$.

The Kendal τ is 0.53986. The parameter θ depending on the copula family is as in the following table.

Table 1: The value of the estimated parameter θ depending on the copula family.

Family	**Constraints on** $\tau \in [-1,1]$	θ
Clayton: $\theta > 0$	$\tau > 0$	2.34646
Frank: $\theta \neq 0$	$\tau \neq 0$	0.00948
Gumbel-Hougaard: $\theta \geq 1$	$\tau \geq 0$	2.17323
Gumbel-Barnett: $0 < \theta \leq 1$	$\tau < 0$	not our case
Ali-Mikhail-Haq: $-1 \leq \theta \leq 1$	$\frac{5-8\ln 2}{3} \leq \tau \leq \frac{1}{3}$	not our case
FGM: $-1 \leq \theta \leq 1$	$\lvert\tau\rvert \leq \frac{2}{9}$	not our case
Fréchet: $\theta \geq 0$	$\tau > 0$	0.53986

In the following graphics there are represented first the above boxes in Ouv and Oxy, and the data points with $U_i = F(Gdl_i)$ and $V_i = G(M_i)$, respectively $X_i = Gdl_i$ and $Y_i = M_i$. We represent also the isolines (23) in the plane Ouv with $k = 2$, $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$, and the corresponding isolines for the plane Oxy. These graphics are represented for each case of copula for which we have estimated the parameter θ (the constraints for the above tables are fulfilled).

In these graphics each class has a code given by four integer numbers: first number is for C_{00}, the second number is for C_{01}, the third number is for C_{10} and the last number is for C_{11}. The number corresponding to a copula type depends on the position of its value and α_i, tacking $\alpha_0 = 0$ and $\alpha_3 = 1$. For instance the code from the fourth position is *k* if $\alpha_k \leq C_{11}(u,v) < \alpha_{k+1}$ for any *u, v* in the considered region. The region from the center (containing the middle of the square) has the code (0,0,0,0), and the class from the corner (1,0) has the code (0,0,2,0). For each of the 553 points (U_i, V_i) obtained by the application of the (normal) cumulative distribution function on Gdl_i and M_i for the clustered market *i* we find the code of its class. When the point is in an old class we count it for this class, and we memorize its class's code if the point is in a new class. In each of the obtained regions we write the class number, and, between parentheses, the number of clustered markets from the involved class.

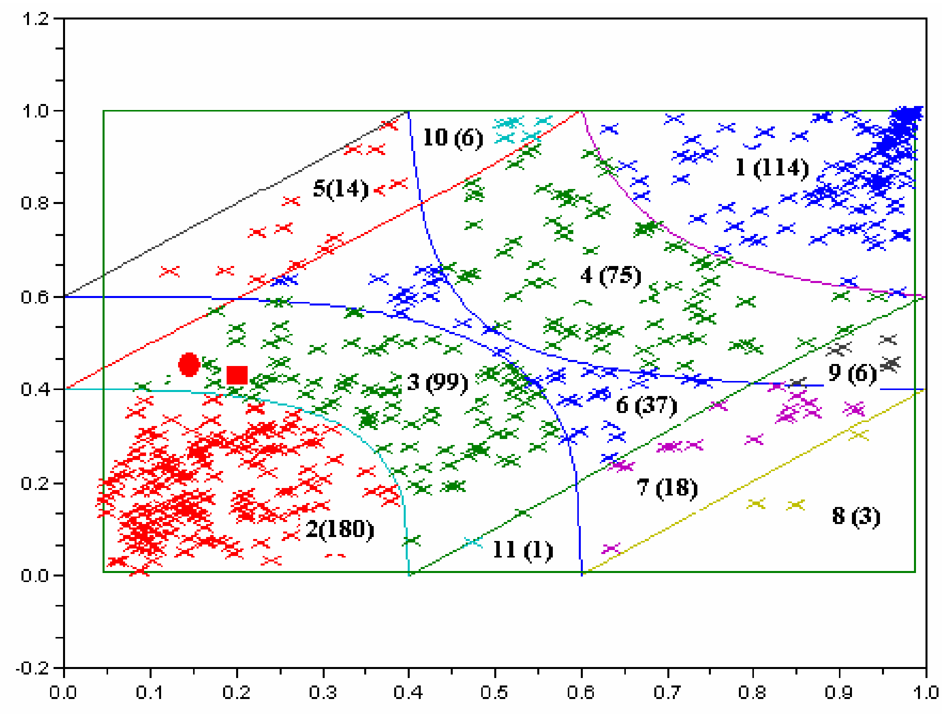

Fig. 1a: The graphics in the coordinates *u, v* in the case of Clayton copula

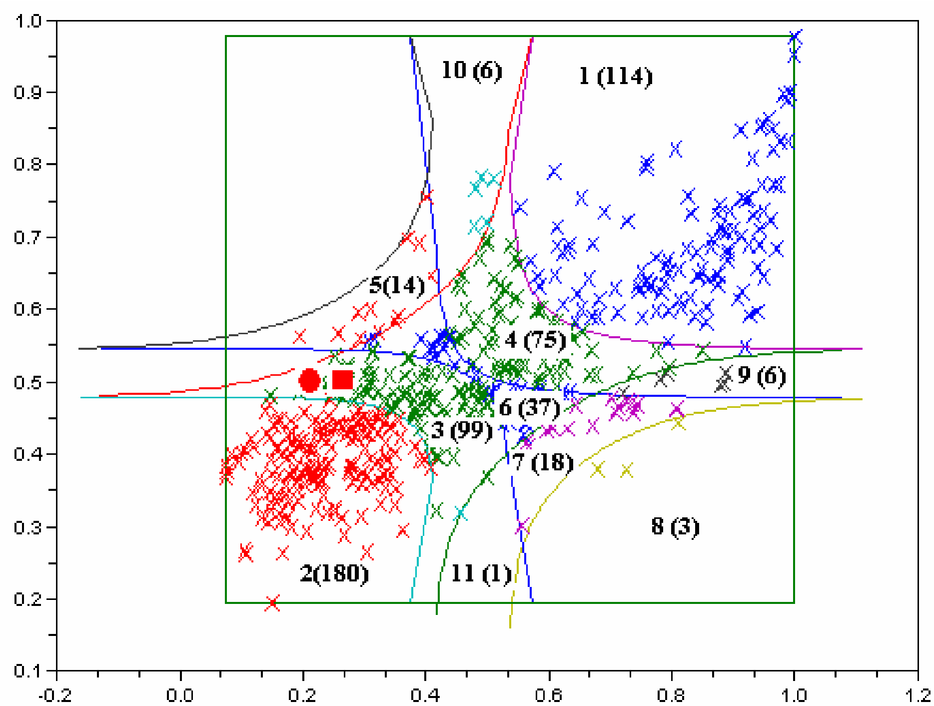

Fig. 1b: The graphics in the coordinates *x, y* in the case of Clayton copula

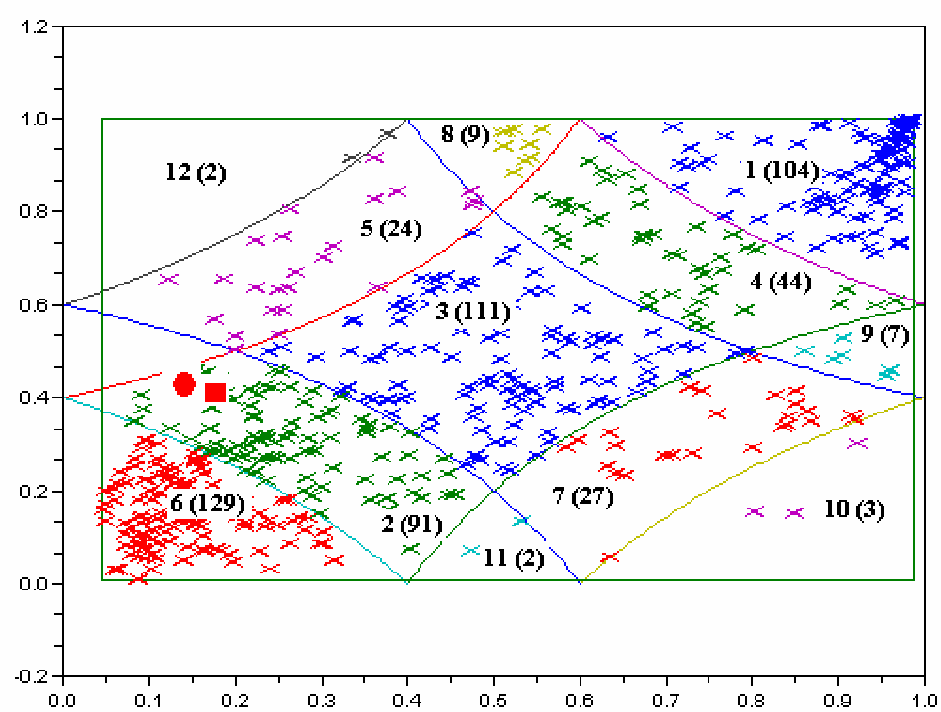

Fig. 2a: The graphics in the coordinates *u, v* in the case of Frank copula

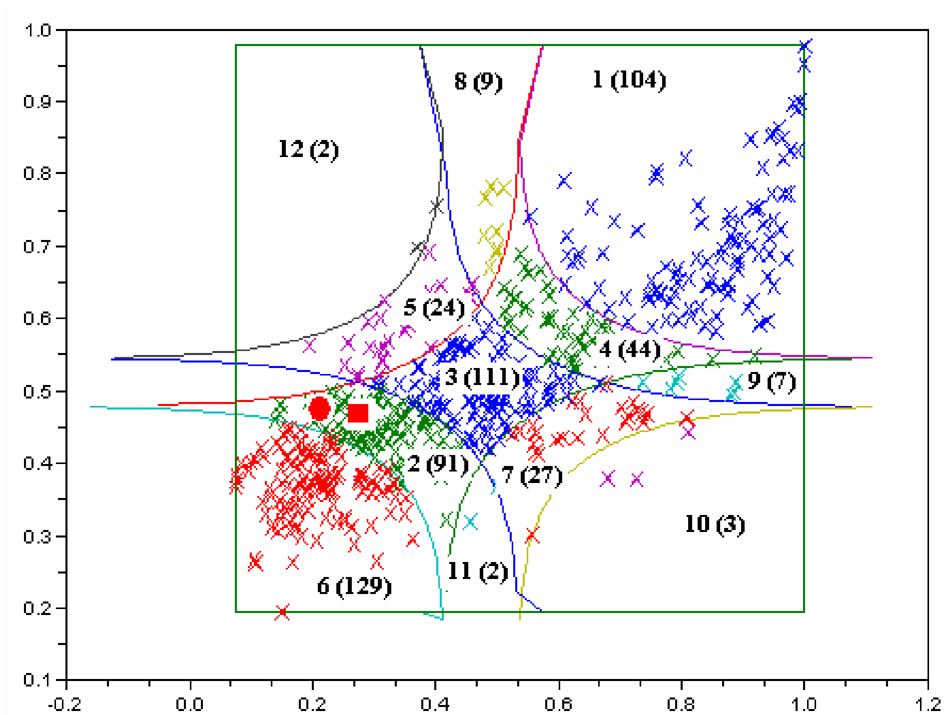

Fig. 2b: The graphics in the coordinates *x, y* in the case of Frank copula

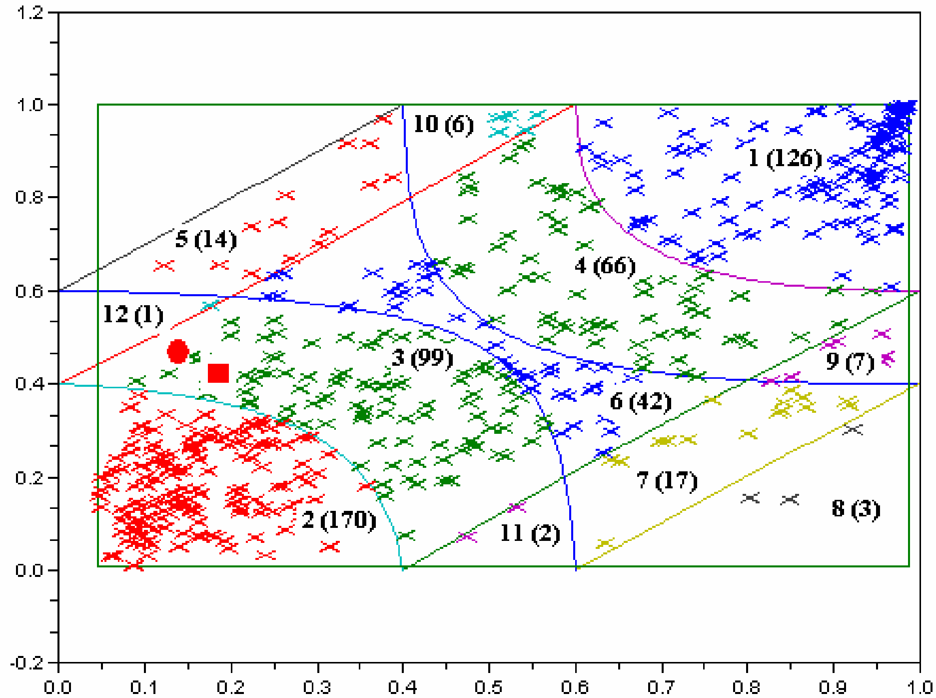

Fig. 3a: The graphics in the coordinates *u, v* in the case of Gumbel-Hougaard copula

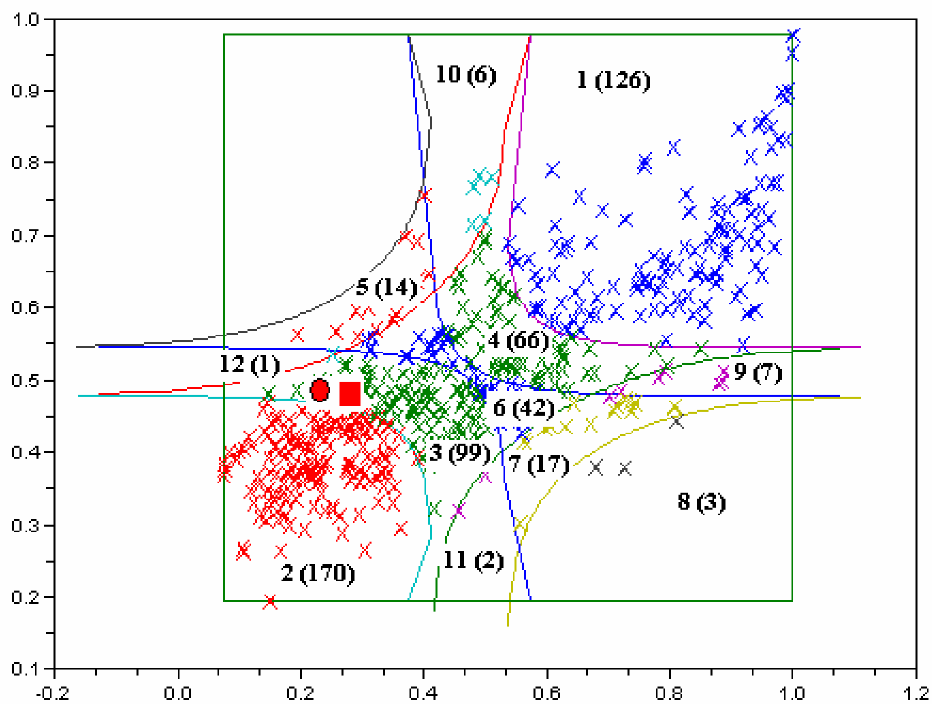

Fig. 3b: The graphics in the coordinates *x, y* in the case of Gumbel-Hougaard copula

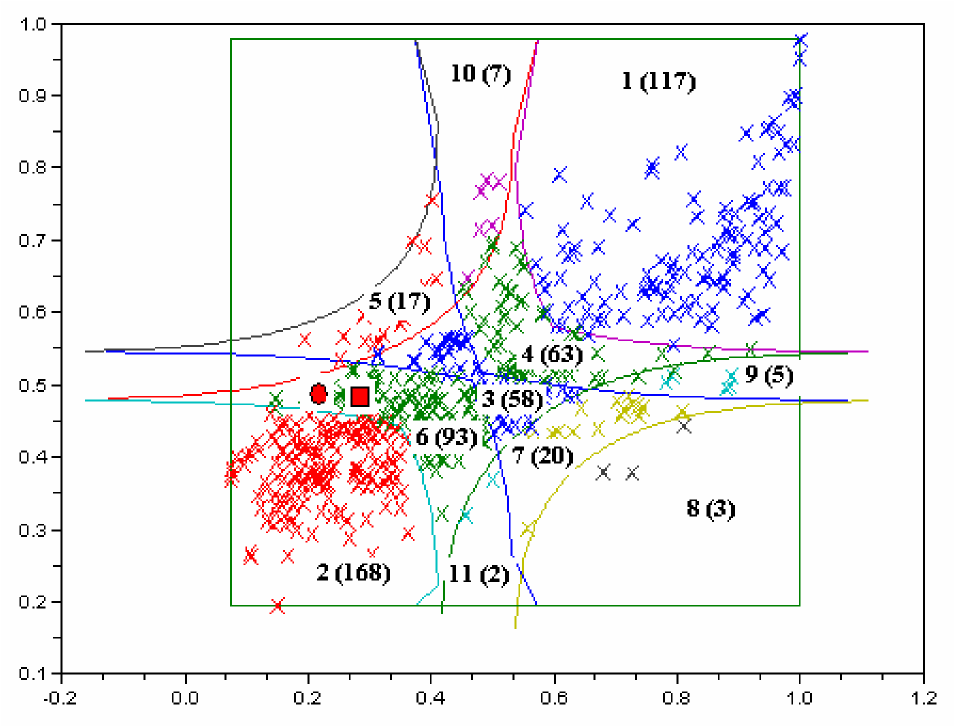

Fig. 4a: The graphics in the coordinates *u, v* in the case of Fréchet copula

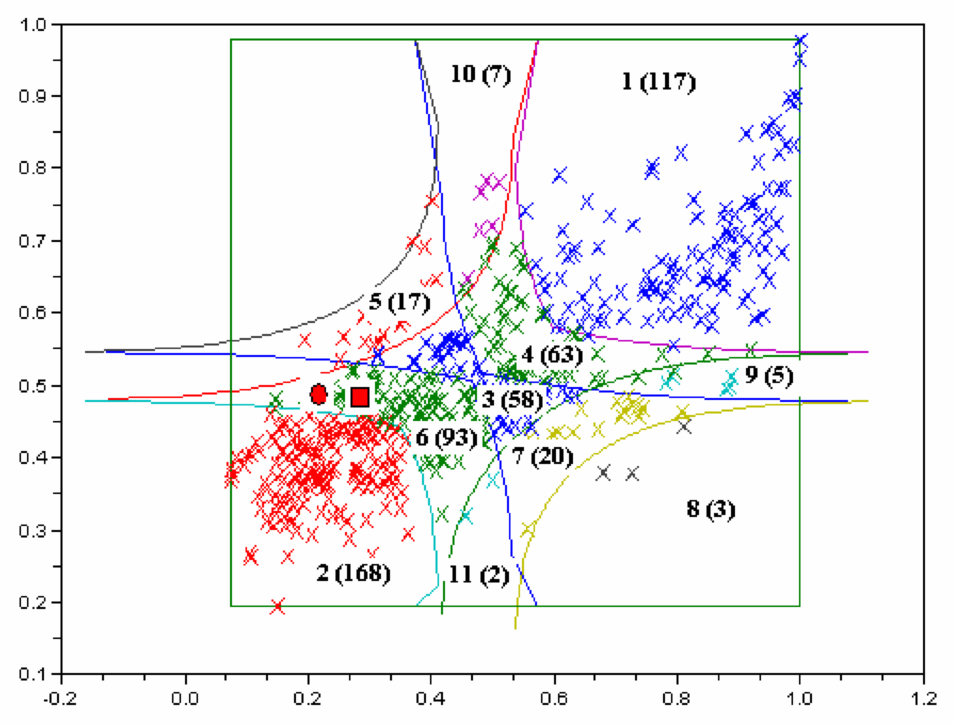

Fig. 4b: The graphics in the coordinates *x, y* in the case of Fréchet copula

We notice that the two national systems from 2004 (represented by a circle in the above graphics, with $Gdl = 0.2287$, $M = 0.4996$, $U = F(Gdl) = 0.1524$ and $V = G(M) = 0.4579$) and 2008 (represented by a square in the above graphics, with $Gdl = 0.2861$, $M = 0.4856$, $U = F(Gdl) = 0.2161$ and $V = G(M) = 0.4191$) are in the same class having the code (1,0,0,0), i.e.

$0.4 \le C_{00}(U,V) < 0.6$, $C_{11}(U,V) < 0.4$, $C_{01}(U,V) < 0.4$, $C_{10}(U,V) < 0.4$.

The numbers of clustered markets in each class depending on the year (2004, 2008 or both) and on the level (group, division, section or all three levels) are listed in Appendix A. The line of the class that contains the two national systems is bolded. The star at the exponent at “Total” means that the total does not contain the national system. Two stars in the last total means that we did not take into account the two national systems. For instance, in the case of Clayton copula the totals are 44^* for the year 2004, 53^* for the year 2008 and 97^{**} for both year. It means that, if we take into account the national systems, these totals would be 45 for the year 2004, 54 for the year 2008 and 99 for both years.

4. Conclusions

Our classification has a probabilistic interpretation: each region obtained by isolines is such that the four probabilities resulting from the four copula types from definition 3 are in given intervals bordered by α_j. It has also more possible classes than the nine regions from the case of Mereuță (see [7]): in each case of copula family there are 13 possible classes. Even the effective number of classes is greater (11 in the cases of Clayton and Fréchet copula, respectively 12 in the cases of Frank and Gumbel-Hougaard copula).

There are also similitudes between the classification of Mereuță and those from this paper. In the classification of Mereuță the regions with small *Gdl* and *M*, and with both values big (the heads of the main diagonal) there are relative big numbers of the contained clustered markets: both numbers are equal to 97. The same thing we can say about our case. For the smallest values of *Gdl* and *M* we obtain 180 clustered markets in the case of Clayton copula, 129 clustered markets in the case of Frank copula, 170 clustered markets in the case of Gumbel-Hougaard copula, respectively 168 clustered markets in the case of Fréchet copula. For the highest values of *Gdl* and *M* we obtain 114 clustered markets in the case of Clayton copula, 104 clustered markets in the case of Frank copula, 126 clustered markets in the case of Gumbel-Hougaard copula, respectively 117 clustered markets in the case of Fréchet copula.

On the secondary diagonal the above numbers are small. The number of clustered markets with high *Gdl* and low *M* (the class from bottom-right corner) is 2 in the case of Mereuță, respectively 3 in the case of this paper, for each case of copula.

The number of clustered markets with low *Gdl* and high *M* (the class from top-left corner) is 5 in the case of Mereuță, 2 in the case of Frank copula, and 0 (no clustered market in the class) in the other cases.

In the case of Frank copula there is also an interesting similitude between our classification and the classification of Mereuță for the middle class (medium *Gdl* and *M*): the number of clustered markets is 110 in the case of Mereuță, and 111 in our case.

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Appendix A
The numbers of clustered markets depending on the year and on the level

Table 2: The number of clustered markets in the case of Clayton copula

Class number	2004				2008				2004 and 2008			
	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total
1	37	7	2	46	53	12	3	68	90	19	5	114
2	51	20	5	76	64	32	8	104	115	52	13	180
3	**37**	**6**	**1**	**44***	**37**	**10**	**6**	**53***	**74**	**16**	**7**	**97****
4	27	7	1	35	26	13	1	40	53	20	2	75
5	2	2	1	5	5	3	1	9	7	5	2	14
6	9	3	2	14	17	6	0	23	26	9	2	37
7	6	1	0	7	9	2	0	11	15	3	0	18
8	2	0	0	2	1	0	0	1	3	0	0	3
9	1	0	0	1	4	1	0	5	5	1	0	6
10	1	1	1	3	2	1	0	3	3	2	1	6
11	1	0	0	1	0	0	0	0	1	0	0	1

Table 3: The number of clustered markets in the case of Frank copula

Class number	2004				2008				2004 and 2008			
	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total
1	31	7	2	40	49	12	3	64	80	19	5	104
2	**32**	**11**	**2**	**45***	**26**	**16**	**2**	**44***	**58**	**27**	**4**	**89****
3	35	8	1	44	48	16	3	67	83	24	4	111
4	19	5	1	25	14	5	0	19	33	10	1	44
5	5	2	2	9	8	4	3	15	13	6	5	24
6	35	12	4	51	49	21	8	78	84	33	12	129
7	9	1	0	10	14	3	0	17	23	4	0	27
8	3	1	1	5	3	1	0	4	6	2	1	9
9	2	0	0	2	4	1	0	5	6	1	0	7
10	2	0	0	2	1	0	0	1	3	0	0	3
11	1	0	0	1	1	0	0	1	2	0	0	2
12	0	0	0	0	1	1	0	2	1	1	0	2

Table 4: The number of clustered markets in the case of Gumbel-Hougaard copula

Class number	2004				2008				2004 and 2008			
	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total
1	42	10	2	54	56	13	3	72	98	23	5	126
2	49	19	5	73	61	28	8	97	110	47	13	170
3	**36**	**6**	**1**	**43***	**35**	**13**	**6**	**54***	**71**	**19**	**7**	**97****
4	23	4	1	28	25	12	1	38	48	16	2	66
5	2	2	1	5	5	3	1	9	7	5	2	14
6	11	4	2	17	18	7	0	25	29	11	2	42
7	6	1	0	7	8	2	0	10	14	3	0	17
8	2	0	0	2	1	0	0	1	3	0	0	3
9	1	0	0	1	5	1	0	6	6	1	0	7

10	1	1	1	3	2	1	0	3	3	2	1	6
11	1	0	0	1	1	0	0	1	2	0	0	2
12	0	0	0	0	1	0	0	1	1	0	0	1

Table 5: The number of clustered markets in the case of Fréchet copula

Class number	2004				2008				2004 and 2008			
	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total	Group level	Division level	Section level	Total
1	37	7	2	46	55	13	3	71	92	20	5	117
2	49	19	5	73	59	28	8	95	108	47	13	168
3	18	6	2	26	23	9	0	32	41	15	2	58
4	25	6	1	32	22	8	1	31	47	14	2	63
5	2	2	1	5	7	4	1	12	9	6	2	17
6	**31**	**5**	**1**	**37**[*]	**35**	**13**	**6**	**54**[*]	**66**	**18**	**7**	**91**[**]
7	6	1	0	7	10	3	0	13	16	4	0	20
8	2	0	0	2	1	0	0	1	3	0	0	3
9	1	0	0	1	3	1	0	4	4	1	0	5
10	2	1	1	4	2	1	0	3	4	2	1	7
11	1	0	0	1	1	0	0	1	2	0	0	2